Analytical Expansions for Parabolic Equations

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Abstract

We consider the Cauchy problem associated with a general parabolic partial differential equation in d dimensions. We find a family of closed-form asymptotic approximations for the unique classical solution of this equation as well as rigorous short-time error estimates. Using a boot-strapping technique, we also provide convergence results for arbitrarily large time intervals.

Keywords

parabolic PDE | asymptotic expansion | singular perturbation | analytical approximation

MSC codes

35K10 | 74G10 | 91G20

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Adjoint Expansions in Local Lévy Models

Stefano Pagliarani , Andrea Pascucci , Candia Riga

Abstract +

We propose a novel method for the analytical approximation in local volatility models with Lévy jumps. The main result is an expansion of the characteristic function in a local Lévy model, which is worked out in the Fourier space by considering the adjoint formulation of the pricing problem. Combined with standard Fourier methods, our result provides efficient and accurate pricing formulae. In the case of Gaussian jumps, we also derive an explicit approximation of the transition density of the underlying process by a heat kernel expansion: the approximation is obtained in two ways, using partial integro-differential equation techniques and working in the Fourier space. Numerical tests confirm the effectiveness of the method.

Closed-Form Asymptotics and Numerical Approximations of 1D Parabolic Equations with Applications to Option Pricing

Wen Cheng , Nick Costanzino , John Liechty , Anna Mazzucato , Victor Nistor

Abstract +

We construct closed-form asymptotic formulas for the Green's function of parabolic equations (e.g., Fokker–Planck equations) with variable coefficients in one space dimension. More precisely, let $u(t,x)=\int\mathcal{G}_t(x,y)f(y)dy$ be the solution of $\partial_tu-(au”+bu'+cu)=0$ for $t>0$, $u(0,x)=f(x)$. Then we find computable approximations $\mathcal{G}_t^{[n]}$ of $\mathcal{G}_t$. The approximate kernels are derived by applying the Dyson–Taylor commutator method that we have recently developed for short-time expansions of heat kernels on arbitrary dimension Euclidean spaces. We then utilize these kernels to obtain closed-form pricing formulas for European call options. The validity of such approximations to large time is extended using a bootstrap scheme. We prove explicit error estimates in weighted Sobolev spaces, which we test numerically and compare to other methods.

The Small-Maturity Heston Forward Smile

Antoine Jacquier , Patrick Roome

Abstract +

In this paper we investigate the asymptotics of forward-start options and the forward implied volatility smile in the Heston model as the maturity approaches zero. We prove that the forward smile for out-of-the-money options explodes and compute a closed-form high-order expansion detailing the rate of the explosion. Furthermore, the result shows that the square-root behavior of the variance process induces a singularity such that for certain parameter configurations one cannot obtain high-order out-of-the-money forward smile asymptotics. In the at-the-money case a separate model-independent analysis shows that the small-maturity limit is well defined for any Itô diffusion. The proofs rely on the theory of sharp large deviations [B. Bercu and A. Rouault, *Theory Probab. Appl.*, 46 (2002), pp. 1--19] (and refinements), and incidentally we provide an example of degenerate large deviation behavior.

Short-Time Expansions for Call Options on Leveraged ETFs Under Exponential Lévy Models with Local Volatility

José E. Figueroa-López , Ruoting Gong , Matthew Lorig

Abstract +

In this article, we consider the small-time asymptotics of options on a *leveraged exchange-traded fund* (LETF) when the underlying exchange-traded fund (ETF) exhibits both local volatility and jumps of either finite or infinite activity. We show that leverage modifies the drift, volatility, jump intensity, and jump distribution of a LETF in addition to inducing the possibility of default, even when the underlying ETF price remains strictly positive. Our main results are closed-form expressions for the leading-order terms of off-the-money European call and put LETF option prices near expiration, with explicit error bounds. These results show that the price of an out-of-the-money European call on a LETF with positive (negative) leverage is asymptotically equivalent, in short time, to the price of an out-of-the-money European call (put) on the underlying ETF, but with modified spot and strike prices. Similar relationships hold for other off-the-money European options. These observations, in turn, suggest a method to hedge off-the-money LETF options near expiration using options on the underlying ETF. Finally, we derive a second-order expansion for the implied volatility of an off-the-money LETF option and show both analytically and numerically how this is affected by leverage.

The Small-Maturity Smile for Exponential Lévy Models

José E. Figueroa-López , Martin Forde

Abstract +

We derive a small-time expansion for out-of-the-money call options under an exponential Lévy model, using the small-time expansion for the distribution function given in [J. Figueroa-López and C. Houdré, *Stochastic Process. Appl.*, 119 (2009), pp. 3862–3889], combined with a change of numéraire via the Esscher transform. In particular, we find that the effect of a nonzero volatility σ of the Gaussian component of the driving Lévy process is to increase the call price by $\frac{1}{2}\sigma^2 t^2 e^{k}\nu(k)(1+o(1))$ as $t \to 0$, where ν is the Lévy density. Using the small-time expansion for call options, we then derive a small-time expansion for the implied volatility $\hat{\sigma}_{t}^{2}(k)$ at log-moneyness *k*, which sharpens the first order estimate $\hat{\sigma}_{t}^{2}(k)\sim \frac{\frac{1}{2}k^2}{t\log(1/t)}$ given in [P. Tankov, *Pricing and hedging in exponential Lévy models: Review of recent results*, in Paris-Princeton Lectures on Mathematical Finance, Springer, Berlin, 2011, pp. 319–359]. Our numerical results show that the second order approximation can significantly outperform the first order approximation. Our results are also extended to a class of time-changed Lévy models. We also consider a small-time, small log-moneyness regime for the CGMY model and apply this approach to the small-time pricing of at-the-money call options; we show that for $Y\in(1,2)$, $\lim_{t\to{}0}t^{-1/Y}\mathbb{E}(S_t-S_0)_{+}=S_{0}\mathbb{E}^{*}(Z_{+})$ and the corresponding at-the-money implied volatility $\hat{\sigma}_t(0)$ satisfies $\lim_{t \to 0}\hat{\sigma}_t(0)/t^{1/Y-1/2}=\sqrt{2\pi}\,\mathbb{E}^{*}(Z_{+})$, where *Z* is a symmetric *Y*-stable random variable under $\mathbb{P}^*$ and *Y* is the usual parameter for the CGMY model appearing in the Lévy density $\nu(x)=C x^{-1-Y}e^{-M x}{\bf 1}_{\{x>0\}}+C |x|^{-1-Y}e^{-G|x|}{\bf 1}_{\{x<0\}}$ of the process.

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