


[Home](#) > [Decisions in Economics and Finance](#) > [Article](#)

Reaching nirvana with a defaultable asset?

| Published: 01 June 2017

| Volume 40, pages 31–52 (2017) [Cite this article](#)[Save article](#)[View saved research](#) >



[Decisions in Economics and Finance](#)

[Aims and scope](#) →[Submit manuscript](#) →[Anna Battauz](#)¹, [Marzia De Donno](#)² & [Alessandro Sbuelz](#)  orscid.org/0000-0002-8602-3603³ 276 Accesses  13 Citations [Explore all metrics](#) →

Abstract

We study the optimal dynamic portfolio exposure to predictable default risk, taking inspiration from the search for yield by means of defaultable assets observed before the 2007–2008 crisis and in its aftermath. Under no arbitrage, default risk is compensated by an ‘yield pickup’ that can strongly attract aggressive investors via an investment-horizon effect in their optimal non-myopic portfolios. We show it by stating the optimal dynamic portfolio problem of Kim and Omberg (Rev Financ Stud 9:141–161, [1996](#)) for a defaultable risky asset and by rigorously proving the existence of nirvana-type solutions. We achieve such a contribution to the portfolio optimization literature by means of a careful, closed-form-yielding adaptation to our defaultable asset setting of the general convex

duality approach of Kramkov and Schachermayer (Ann Appl Probab 9(3):904–950, [1999](#); Ann Appl Probab 13(4):1504–1516, [2003](#)).

 This is a preview of subscription content, [log in via an institution](#)  to check access.

Access this article

[Log in via an institution](#) →

Subscribe and save

Springer+

from €37.37 /Month

- Starting from 10 chapters or articles per month
- Access and download chapters and articles from more than 300k books and 2,500 journals
- Cancel anytime

[View plans](#) →

Buy Now

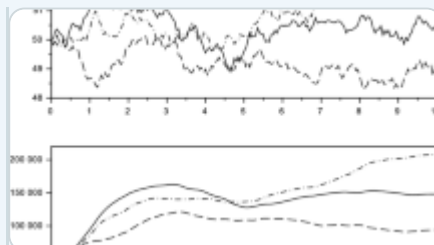
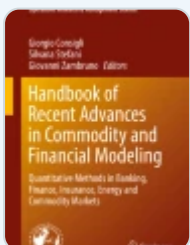
[Buy article PDF 39,95 €](#)

Price includes VAT (Poland)

Instant access to the full article PDF.

[Institutional subscriptions](#) →

Similar content being viewed by others



Portfolio choice, portfolio

Explore related subjects

Discover the latest articles, books and news in related subjects, suggested using machine learning.

[Behavioral Finance](#)

[Capital Markets](#)

[Mathematical Finance](#)

[Microfinance](#)

[Risk Theory](#)

[Calculus of Variations and Optimization](#)

[Dynamic Portfolio Optimization in Financial Markets](#)

Notes

1. Cox and Huang ([1989](#)) require the global Lipschitz continuity of the diffusive coefficient for the risky asset-value process [see Conditions A and B at p. 46 in Cox and Huang ([1989](#))]. By contrast, the diffusive coefficient in our setting is the square root of the risky asset value.
2. See, for instance, Davydov and Linetsky ([2001](#)), p. 952, first paragraph, with $(S_t = P_t)$ and $(p = \frac{1}{2})$.
3. The objective probability of the asset defaulting within the date $(T > 0)$ is $(\mathbb{P} \left[P_h = 0, 0 \leq h \leq T \quad P_0 = p \right]) = \Gamma \left(\frac{2 \left(r + \xi \right) p}{1 - e^{-\left(r + \xi \right) T}}, 1 \right)$, where $(\Gamma(k) = \int_k^{+\infty} u^{k-1} e^{-u} du)$ ($k \geq 0$) is the incomplete gamma function [see, e.g., the Proposition 1 in Campi and Sbuelz ([2005](#))].
4. The nonnegativity requirement is innocuous in our setting as the utility function U is $(-\infty)$ for negative wealth levels.

5. Battauz et al. (2015) apply the Kramkov and Schachermayer (1999, 2003) approach to the standard Kim and Omberg (1996) portfolio problem with a non-defaultable risky asset.
6. There are parameter values that make the investor with $(\phi > 1)$ take a net short position in the risky defaultable asset.
7. Similar investment-horizon effects have been found in the literature on dynamic portfolio choice with a risky non-defaultable asset characterized by a mean-reverting drift and a constant volatility [see, for example, Koijen et al. (2009)].
8. The functions U and V are conjugate if and only if $(U(w) = \inf_{y > 0} (V(y) + wy))$ and $(V(y) = \sup_{w > 0} (U(w) - wy))$.
9. The superscript $(\left(\cdot \right)^{\{DS\}})$ refers to the notations of Delbaen and Shirakawa (2002), whereas a , b , c are defined in the statement of Proposition 3.2.

References

Battauz, A., De Donno, M., Sbuelz, A.: Kim and Omberg revisited: the duality approach. *J. Probab. Stat.* **2015**, Article ID 581854, p. 6 (2015).

doi:[10.1155/2015/581854](https://doi.org/10.1155/2015/581854)

Becker, B., Ivashina, V.: Reaching for yield in the bond market. *J. Finance* **70**, 1863–1902 (2015)

Bekaert, G., Ang, A.: International asset allocation with regime shifts. *Rev.*

Financ. Stud. **15**(4), 1137-1187 (2002)

Branger, N., Larsen, L.S., Munk, C.: Robust portfolio choice with ambiguity and learning about return predictability. *J. Bank. Finance* **37**(5), 1397-1411 (2013)

Buch, C., Eickmeier, S., Prieto, E.: Macroeconomic factors and microlevel bank behavior. *J. Money Credit Bank.* **46**(4), 715-751 (2014)

Campi, L., Sbuelz, A.: Closed-form pricing of benchmark equity default swaps under the CEV assumption. *Risk Lett.* **1**, ISSN: 1740-9551 (2005)

Chodorow-Reich, G.: Effects of unconventional monetary policy on financial institutions. *Brookings Papers on Economic Activity* (Spring), pp. 155-204 (2014)

Cox, J.C., Ross, S.: The valuation of options for alternative stochastic processes. *J. Financ. Econ.* **3**, 145-166 (1976)

Cox, J.C., Huang, C.F.: Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econ. Theory* **49**, 33-83 (1989)

Davydov, D., Linetsky, V.: Pricing and hedging path-dependent options under the CEV process. *Manage. Sci.* **47**(7):949-965 (2001).

doi:[10.1287/mnsc.47.7.949.9804](https://doi.org/10.1287/mnsc.47.7.949.9804)

Delbaen, F., Shirakawa, H.: A note on option pricing for the constant elasticity of variance model. *Financ. Eng. Jpn. Mark.* **9**(2), 85-99 (2002)

Della Corte, P., Sarno, L., Tsiakas, I.: Volatility and correlation timing in active currency management. In: James, J., Marsh, I.W., Sarno, L. (eds.) *Handbook of*

Exchange Rates. Wiley, Hoboken NJ, USA (2012).

doi:[10.1002/9781118445785.ch15](https://doi.org/10.1002/9781118445785.ch15)

Dell’Ariccia, G., Laeven, L., Marquez, R.: Monetary policy, leverage, and bank risk-taking. *J. Econ. Theory* **149**, 65–99 (2014)

Detemple, J.B., Garcia, R., Rindisbacher, M.: A Monte-Carlo method for optimal portfolios. *J. Finance* **58**, 401–446 (2003)

Detemple, J.B., Garcia, R., Rindisbacher, M.: Intertemporal asset allocation: a comparison of methods. *J. Bank. Finance* **29**(11), 2821–2848 (2005)

Diamond, D.W., Rajan, R.G.: The credit crisis: conjectures about causes and remedies. *Am. Econ. Rev.* **99**(2), 606–610 (2009)

Di Maggio, M.: Market Turmoil and Destabilizing Speculation. Columbia Working Paper (2013)

Di Maggio, M., Kacperczyk, M.T.: The unintended consequences of the zero lower bound policy. *J. Financ. Econ. (JFE)* (2016). doi:[10.2139/ssrn.2458587](https://doi.org/10.2139/ssrn.2458587)
(forthcoming)

Gennaioli, N., Martin, A., Rossi, S.: Sovereign default. Domestic banks, and financial institutions. *J. Finance* **69**(2), 819–866 (2014)

Gollier, C.: Discounting an uncertain future. *J. Public Econ.* **85**, 149–166 (2002)

Gozzi, F., Russo, F.: Verification theorems for stochastic optimal control problems via a time dependent Fukushima–Dirichlet decomposition. *Stoch. Process. Appl.* **116**(11), 1530–1562 (2006)

Guidolin, M., Timmermann, A.: Asset allocation under multivariate regime switching. *J. Econ. Dyn. Control* **31**, 3503–3544 (2007)

Guidolin, M., Hyde, S.: Can VAR models capture regime shifts in asset returns? A long-horizon strategic asset allocation perspective. *J. Bank. Finance* **36**(3), 695–716 (2012)

Ioannidou, V., Ongena, S., Peydro, J.-L.: Monetary policy, risk-taking, and pricing: evidence from a quasi-natural experiment. *Rev Finance* **19**, 95–144 (2015)

Jiménez, G., Ongena, S., Peydró, J.L., Saurina, J.: Hazardous times for monetary policy: what do twenty-three million bank loans say about the effects of monetary policy on credit risk? *Econometrica* **82**(2), 463–505 (2014)

Kacperczyk, M., Schnabl, P.: How safe are money market funds? *Q. J. Econ.* **128**, 1073–1122 (2013)

Kim, T.S., Omberg, E.: Dynamic nonmyopic portfolio behavior. *Rev. Financ. Stud.* **9**, 141–161 (1996)

Klebaner, F., Lipster, R.: When a stochastic exponential is a true martingale. Extension of a method of benes. *Theory Probab. Appl.* **58**(1), 38–62 (2014)

Koijen, R.S.J., Rodríguez, J.C., Sbuelz, A.: Momentum and mean reversion in strategic asset allocation. *Manag. Sci.* **55**, 1199–1213 (2009)

Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete market. *Ann. Appl. Probab.* **9**(3), 904–950 (1999)

Kramkov, D., Schachermayer, W.: Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.* **13**(4), 1504–1516 (2003)

Larsen, L.S., Munk, C.: The costs of suboptimal dynamic asset allocation: general results and applications to interest rate risk, stock volatility risk, and growth/value tilts. *J. Econ. Dyn. Control* **36**(2012), 266–293 (2012)

Lioui, A., Poncet, P.: International asset allocation: a new perspective. *J. Bank. Finance* **27**(11), 2203–2230 (2003)

Liu, J.: Portfolio selection in stochastic environments. *Rev. Financ. Stud.* **20**, 1–39 (2007)

Martin, I.: On the valuation of long-dated assets. *J. Polit. Econ.* **120**(2), 346–358 (2012)

Merton, R.C.: Optimum consumption and portfolio rules in a continuous-time model. *J. Econ. Theory* **3**, 373–413 (1971)

Nielsen, L.T., Vassalou, M.: The instantaneous capital market line. *Econ. Theor.* **28**, 651–664 (2006)

Rajan, R.: Has Financial Development Made the World Riskier? Proceedings, Federal Reserve Bank of Kansas City, August, pp. 313–69 (2005)

Wachter, J.: Portfolio and consumption decisions under mean-reverting returns: an exact solution for complete markets. *J. Financ. Quant. Anal.* **37**, 63–91 (2002)

Weitzman, M.L.: Why the far-distant future should be discounted at its lowest

possible rate. *J. Environ. Econ. Manag.* **36**, 201–208 (1998)

Weitzman, M.L.: On modeling and interpreting the economics of catastrophic climate change. *Rev. Econ. Stat.* **91**, 1–19 (2009)

Acknowledgements

We would like to thank the editors and two anonymous referees for their insightful comments and suggestions.

Author information

Authors and Affiliations

Department of Finance and IGIER, Bocconi University, Milan, Italy

Anna Battauz

Department of Economics, University of Parma, Parma, Italy

Marzia De Donno

Department of Mathematical Sciences, Mathematical Finance and Econometrics, Catholic University of Milan, Largo Gemelli, 1, 20123, Milan, Italy

Alessandro Sbuelz

Corresponding author

Correspondence to [Alessandro Sbuelz](#).

A Appendix

1.1 A.1 Proof of proposition [3.1](#)

The problem of utility maximization can be written as $J(w) = \left(e^{\rho T} \right)^{1-\phi} u(w)$ where u is defined as

$$u(w) = \sup_{\tilde{W} \in \mathcal{W}} E[U(\tilde{W}_T)]. \quad (A.1)$$

We apply to problem (A.1) the duality approach developed by Kramkov and Schachermayer (1999, 2003). To this aim, we observe that the utility function U satisfies Inada conditions [equation (2.4) in Kramkov and Schachermayer (1999)]. Let V denote the conjugate function⁸ of U , that is $V(y) = \frac{\phi}{1-\phi} y^{-\frac{1-\phi}{\phi}}$ and define

$$v(y) = E[V(\eta)]$$

where (η) is given by (2.3). Kramkov and Schachermayer (2003) show that if $v(y) < \infty$ for all $(y > 0)$, then $u(w) < \infty$ for all $(w > 0)$ and u and v are conjugate. They also prove that the optimal solution $(\tilde{W}^*) \in \mathcal{W}(w)$ to (A.1) exists and is unique. Moreover, taking $(y = u'(w))$ (or equivalently $(w = -v'(y))$), they provide the dual relation for the optimizer $(\tilde{W}^*) = -V'(y\eta)$ (see Theorems 1,2 and Note 3).

Assuming that $(E[\eta^{-\frac{1-\phi}{\phi}}] < +\infty)$, from the condition $(w = -v'(y))$, we get

$$y = \left(\frac{w}{E[\eta^{-\frac{1-\phi}{\phi}}]} \right)^{\frac{1-\phi}{\phi}} \eta$$

and

$$\tilde{W}^* = -V'(y\eta) = \left(\left(\frac{w}{E[\eta^{-\frac{1-\phi}{\phi}}]} \right)^{\frac{1-\phi}{\phi}} \eta \right)^{-\frac{1-\phi}{\phi}} = w \frac{\eta^{-\frac{1-\phi}{\phi}}}{E[\eta^{-\frac{1-\phi}{\phi}}]}$$

The value function is then

$$\begin{aligned}
 J(w) &= \left(e^{rT} \right)^{1-\phi} E \left[U(\tilde{W}^*) \right] = U(w e^{rT}) E \left[\left(\frac{\eta^{-\frac{1}{\phi}}}{E \left[\eta^{-\frac{1-\phi}{\phi}} \right]} \right)^{1-\phi} \right] \\
 &= U(w e^{rT}) \left(E \left[\eta^{-\frac{1-\phi}{\phi}} \right] \right)^{\phi} \\
 &= U(w e^{rT}) \left(E \left[\exp \left(\frac{1-\phi}{\phi} \int_0^T \sqrt{Y_t} dZ_t + \frac{1-\phi}{2\phi} \int_0^T Y_t dt \right) \right] \right)^{\phi} \\
 &= U(w e^{rT}) F(T, y). \end{aligned}$$

(square)

1.2 A.2 Proof of proposition [3.2](#)

Since $a = \frac{1-\phi}{\phi}$, then

$$\begin{aligned}
 & E \left[\exp \left(\frac{1-\phi}{\phi} \int_0^T \sqrt{Y_t} dZ_t + \frac{1-\phi}{2\phi} \int_0^T Y_t dt \right) \right] \\
 & \quad = E \left[\exp \left(a \int_0^T \sqrt{Y_t} dZ_t + \frac{a}{2} \int_0^T Y_t dt \right) \right]
 \end{aligned}$$

We can write:

$$\begin{aligned}
 & E \left[\exp \left(a \int_0^T \sqrt{Y_t} dZ_t + \frac{a}{2} \int_0^T Y_t dt \right) \right] \\
 & \quad = E \left[L_T \exp \left(\frac{a^2 + a}{2} \int_0^T Y_t dt \right) \right]
 \end{aligned}$$

where

$$L_T = \exp \left(a \int_0^T \sqrt{Y_t} dZ_t - \frac{a^2 + a}{2} \int_0^T Y_t dt \right).$$

(A.2)

The random variable (L_T) in [\(A.2\)](#) is the Radon–Nikodym density of a probability measure equivalent to (\mathbb{P}) . In fact, Theorem⁹ 2.3 in

Delbaen and Shirakawa (2002) applied to $(S^{\text{DS}}=Y, \rho^{\text{DS}}=0.5, r^{\text{DS}}=b>0, \sigma^{\text{DS}}=2\sqrt{c\phi}, \mu^{\text{DS}}=b+2a\sqrt{c\phi}, \theta^{\text{DS}}=a)$ implies that $(\eta_{\cdot T}^{\text{DS}}=L_{\cdot T})$ is the Radon-Nikodym density of an equivalent probability measure $(\hat{\mathbb{Q}})$ equivalent to (\mathbb{P}) . Girsanov's theorem implies then that

$$\begin{aligned} \hat{Z}_t &= Z_t - \int_0^t a \sqrt{Y_s} ds \\ \end{aligned}$$

(A.3)

is a $(\hat{\mathbb{Q}})$ -Brownian motion. Thus,

$$\begin{aligned} F(T, y) &= \left(E^{\hat{\mathbb{Q}}} \left[\exp \left(\frac{a^2 + a}{2} \int_0^T Y_t dt \right) \right] \right)^{\phi}, \\ \end{aligned}$$

(A.4)

where (Y_t) has the following dynamics under $(\hat{\mathbb{Q}})$

$$\begin{aligned} dY_t &= bY_t dt + 2\sqrt{c\phi Y_t} d\hat{Z}_t, \\ \end{aligned}$$

(A.5)

with $(Y_0=y)$. We specify that, if we define the default time $(\tau_y = \inf \{t \geq 0: Y_t=0\})$, we have $(Y_t \equiv 0)$ on $(\tau_y \leq t)$ and Y satisfies the stochastic differential Eq. (A.5) on the whole time interval $(\left[0, T\right])$. To compute the expectation in (A.4), we define the process

$$\begin{aligned} M_t &= \exp \left(\frac{a^2 + a}{2} \int_0^t Y_s ds \right) G(T-t, Y_t) \\ \end{aligned}$$

(A.6)

where $G(t, y)$ is a $(\mathcal{C}^{1,2})$ function to be determined in such a way that $(G(0,y)=1)$ and M is a $(\hat{\mathbb{Q}})$ -martingale. In particular, Eq. (A.6) implies for $(t=0)$ that $(M_0=G(T,y))$ and for $(t=T)$

$$\begin{aligned} M_T &= \exp \left(\frac{a^2 + a}{2} \int_0^T Y_t dt \right) G(0, Y_T) \\ &= \exp \left(\frac{a^2 + a}{2} \int_0^T Y_t dt \right), \end{aligned}$$

since $(G(0, \cdot) = 1)$. The martingality condition $(M_0 = E^{\{\hat{\mathbb{Q}}\}}[M_T])$ yields then

$$G(T,y) = E^{\{\hat{\mathbb{Q}}\}} \left[\exp \left(\frac{a^2 + a}{2} \int_0^T Y_s dt \right) \right],$$

that allows us to find (A.4). By imposing 0 drift on the Ito decomposition under $(\hat{\mathbb{Q}})$ of the process (M_t) we get the partial differential equation for G

$$\begin{aligned} \left\{ \begin{array}{l} G_t = \frac{\xi^2}{2} y G_{yy} + by G_y + \frac{a^2 + a}{2} y G \\ G(0,y) = 1. \end{array} \right. \end{aligned}$$

(A.7)

We guess a solution of the form $(G(t,y) = e^{yg(t)})$ and we obtain the following differential equation for g :

$$\begin{aligned} \left\{ \begin{array}{l} g'(t) = \frac{\xi^2}{2} g^2(t) + bg(t) + \frac{a^2 + a}{2} \\ g(0) = 0. \end{array} \right. \end{aligned}$$

(A.8)

Equation (A.8) is a Riccati equation whose solution is

$$g(t) = \frac{(a^2 + a)(e^{\sqrt{q}t} - 1)}{2}$$

$$\{\sqrt{q}+b+e^{\sqrt{q}t}\left(\sqrt{q}-b\right)\}. \end{aligned}$$$$

Since (M_t) has 0 drift in the Ito decomposition under $(\hat{\mathbb{Q}})$, (M_t) is a $(\hat{\mathbb{Q}})$ local martingale. To conclude that (M_t) is a martingale we define

$$\begin{aligned} z_t = \frac{M_t}{M_0}, \end{aligned}$$$$

which is a $(\hat{\mathbb{Q}})$ local martingale as well, and show that (z_t) is a $(\hat{\mathbb{Q}})$ martingale. To this aim, we first observe that process (z_t) is a stochastic exponential. In fact, Ito formula implies that

$$\begin{aligned} dM_t = e^{\frac{a^2+a}{2}\int_0^t Y_s ds} \frac{\partial}{\partial y} G(T-t, Y_t) 2\sqrt{c\phi(Y_t)} d\hat{Z}_t \end{aligned}$$$$

from the dynamics of Y with respect to $(\hat{\mathbb{Q}})$ in Eq. (A.5). Since $\frac{\partial}{\partial y} G(T-t, Y_t) = e^{Y_t g(T-t)} \cdot g(T-t) = G(T-t, Y_t) \cdot g(T-t)$, we obtain

$$\begin{aligned} dM_t = e^{\frac{a^2+a}{2}\int_0^t Y_s ds} G(T-t, Y_t) \cdot g(T-t) 2\sqrt{c\phi(Y_t)} d\hat{Z}_t \end{aligned}$$$$

leading to

$$\begin{aligned} dM_t = 2\sqrt{c\phi(Y_t)} M_t g(T-t) d\hat{Z}_t. \end{aligned}$$$$

(A.9)

Therefore,

$$\begin{aligned} z_t = 1 + \int_0^t z_s dm_s \end{aligned}$$$$

with

$$\begin{aligned} m_t &= \int_0^t 2\sqrt{c\phi} g(T-s)\sqrt{Y_s} d\hat{Z}_s. \end{aligned}$$

We apply Theorem 4.1 in Klebaner and Lipster (2014) to conclude that $(\frac{z_t}{M_t})$ is a true martingale. In particular, with Klebaner and Lipster notations (4.2) at page 44

$$\begin{aligned} a_s(y) &= by \quad b_s(y) = 2\sqrt{c\phi} \sqrt{y} \quad \text{from (A.5)} \\ \sigma_s(y) &= 2\sqrt{c\phi} \sqrt{y} \quad \text{from our def. of } m_t, \end{aligned}$$

we get

$$\begin{aligned} L_s(y) &= 2ya_s(y) + (b_s(y))^2 \\ &= 2by^2 + 4c\phi y \quad \mathcal{L}_s(y) = 2y \left[a_s(y) + b_s(y)\sigma_s(y) + (b_s(y))^2 \right] \\ &= 2by^2 + 8c\phi y^2 \end{aligned}$$

Since g is bounded, it follows that $(\sigma_s(y))^2$, $(L_s(y))$ and $(\mathcal{L}_s(y))$ are all dominated by a quadratic polynomial in y , and therefore, assumptions (1)-(2)-(3) of Theorem 4.1 are satisfied. This allows us to conclude that $(\frac{z_t}{M_t})$ is a martingale and therefore (M_t) is a martingale as well. Hence we can write (A.4) as

$$\begin{aligned} F(T,y) &= \left(\exp \left(y \frac{(a^2+a)(e^{\sqrt{q}T}-1)}{\sqrt{q}+b+e^{\sqrt{q}T}} \right) \right)^{\phi} \\ &= \exp \left(y \frac{a(e^{\sqrt{q}T}-1)}{\sqrt{q}+b+e^{\sqrt{q}T}} \right) \end{aligned}$$

since

$$\phi(a^2+a) = a.$$

(\square)

1.3 A.3 Proof of proposition [3.3](#)

In what follows, we will mainly work under the martingale measure (\mathbb{Q}, \mathbb{P}) whose density with respect to (\mathbb{P}) is (η) in Eq. [\(2.3\)](#). We denote with $(Z_t^\mathbb{Q})$ the (\mathbb{Q}) -Brownian motion

$$\begin{aligned} Z_t^\mathbb{Q} &= Z_t + \int_0^t \sqrt{Y_s} ds. \end{aligned}$$

(A.10)

Before proving the result, we first list some technical lemmas.

Lemma A.1

Let $(L^* = \frac{L_T}{\eta})$ where (L_T) is given by [\(A.2\)](#). Then $(L_t^* = E^\mathbb{Q}[\frac{L_T}{\eta} | \mathcal{F}_t])$ satisfies the stochastic differential equation

$$\begin{aligned} dL_t^* &= (a+1)L_t^* \sqrt{Y_t} dZ_t^\mathbb{Q} \\ &= \frac{1}{\phi} L_t^* \sqrt{Y_t} dZ_t^\mathbb{Q} \end{aligned}$$

(A.11)

with the initial condition $(L_0^* = 1)$. In particular, (L^*) is the Radon-Nikodym density of the probability measure $(\hat{\mathbb{Q}})$ (whose density with respect to (\mathbb{P}) is L in [\(A.2\)](#) with respect to (\mathbb{Q}) and (\hat{Z}_t) defined in [\(A.3\)](#) can be written as

$$\begin{aligned} \hat{Z}_t &= Z_t^\mathbb{Q} - \frac{1}{\phi} \int_0^t \sqrt{Y_s} ds. \end{aligned}$$

Proof

It is easy to observe that

$$\begin{aligned} L^* &= \frac{L_T}{\eta} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} \end{aligned}$$

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}, \quad \text{\end{aligned}}{\$}$$

and from the definitions of η in (2.3) and of L in (A.2) that

$$\begin{aligned} L_T^* &= \frac{L_T}{\eta} = \exp \left(a \int_0^T \sqrt{Y_s} dZ_s - \frac{a^2}{2} \int_0^T Y_s ds + \int_0^T \sqrt{Y_s} dZ_s + \frac{1}{2} \int_0^T Y_s ds \right) \\ &= \exp \left(\left(a+1 \right) \int_0^T \sqrt{Y_s} dZ_s + \frac{1-a^2}{2} \int_0^T Y_s ds \right). \end{aligned}$$

From the definition of $Z_s^{\mathbb{Q}}$ in Eq. (A.10) we get

$$\begin{aligned} L_T^* &= \exp \left(\left(a+1 \right) \int_0^T \sqrt{Y_s} dZ_s^{\mathbb{Q}} - \frac{1-a^2}{2} \int_0^T Y_s ds \right) \\ &= \exp \left(\left(a+1 \right) \int_0^T \sqrt{Y_s} dZ_s^{\mathbb{Q}} - \frac{\left(a+1 \right)^2}{2} \int_0^T Y_s ds \right). \end{aligned}$$

This is equivalent to

$$dL_t^* = (a+1)L_t^* \sqrt{Y_t} dZ_t^{\mathbb{Q}}. \quad \text{\end{aligned}}{\$}$$

Moreover, from the definition of \hat{Z}_t in (A.3) we get

$$\begin{aligned} \hat{Z}_t &= Z_t - \int_0^t a \sqrt{Y_s} ds \\ &= Z_t^{\mathbb{Q}} - \int_0^t \sqrt{Y_s} ds - \int_0^t a \sqrt{Y_s} ds \\ &= Z_t^{\mathbb{Q}} - \left(a+1 \right) \int_0^t \sqrt{Y_s} ds \quad \left(\text{from (A.10)} \right) \\ &= Z_t^{\mathbb{Q}} - \left(a+1 \right) \int_0^t \sqrt{Y_s} ds \quad \left(\text{with } a+1 = \frac{1}{\phi} \right), \end{aligned}$$

that proves the lemma. \square

Lemma A.2

Let (M_t) be the $(\hat{\mathbb{Q}})$ -martingale defined in (A.6), namely

$$\begin{aligned} M_t &= e^{\frac{a^2 + a}{2} \int_0^t Y_s ds} e^{Y_t g(T-t)}. \end{aligned}$$

Then we have

$$\begin{aligned} dM_t &= 2\sqrt{c\phi Y_t} M_t g(T-t) d\hat{Z}_t \\ &= 2\sqrt{c\phi Y_t} M_t g(T-t) \left(dZ_t^{\mathbb{Q}} - \frac{1}{\phi} \sqrt{Y_t} dt \right) \end{aligned}$$

(A.12)

where (\hat{Z}_t) is defined in (A.3) and $(Z_t^{\mathbb{Q}})$ in Eq. (A.10).

Proof of the Lemma

The first line in Eq. (A.12) is Eq. (A.9), that leads to (A.12) by recalling that

$$\hat{Z}_t = Z_t^{\mathbb{Q}} - \frac{1}{\phi} \int_0^t \sqrt{Y_s} ds.$$

\square

Proof of proposition 3.3

The discounted optimizer

$$\tilde{W}^* = w \frac{\eta^{-\frac{1}{\phi}}}{\eta^{-\frac{1-\phi}{\phi}}} \{ E \left[\eta^{-\frac{1-\phi}{\phi}} \right] \}$$

is the value at time T of a self-financing discounted portfolio, which admits the following representation under (\mathbb{Q})

$$\tilde{W}_t^* = w + \int_0^T \psi_t^* \frac{d\tilde{P}_t}{\tilde{P}_t} = w + \int_0^T \frac{\xi \psi_t}{\tilde{P}_t}$$

$$\begin{aligned} & \left. \frac{F_t}{E_t} \right] \\ & L_t^* M_t \end{aligned} = \frac{w}{G(T,y)} L_t^* M_t \end{aligned} $$$$$

because $(G(T,y) = \left(F(T,y) \right)^{\frac{1}{\phi}} = E_t \left(\eta^{-\frac{1-\phi}{\phi}} \right)^{\frac{1}{\phi}})$. It follows that the differential of the (\mathbb{Q}) -martingale (\tilde{W}_t^*) is given by

$$\begin{aligned} & d\tilde{W}_t^* = \frac{w}{G(T,y)} d\left(L_t^* M_t \right) \\ & = \frac{w}{G(T,y)} L_t^* M_t \left[\frac{1}{\phi} + 2\sqrt{c\phi} g(T-t) \right] \sqrt{Y_t} dZ_t^{\mathbb{Q}} \quad \text{by (A.11) and (A.12)} \\ & = \tilde{W}_t^* \left[\frac{1}{\phi} + 2\sqrt{c\phi} g(T-t) \right] \sqrt{Y_t} dZ_t^{\mathbb{Q}} \end{aligned} $$$$$

Comparing this equation with Eq. (A.13), we obtain

$$\frac{\xi_t \psi_t^*}{\sqrt{Y_t}} = W^*(t) \left[\frac{1}{\phi} + 2\sqrt{c\phi} g(T-t) \right] \sqrt{Y_t} \end{aligned} $$$$$

hence, recalling that $(2\sqrt{c\phi} = \xi)$ and $(g(T-t) = \frac{1}{Y_t} \ln G(T-t, Y_t) = \frac{1}{\phi Y_t} \ln F(T-t, Y_t))$, we have:

$$\begin{aligned} & \psi_t^* = \frac{\tilde{W}_t^*}{\xi} \left[\frac{1}{\phi} + 2\sqrt{c\phi} g(T-t) \right] Y_t \\ & = \tilde{W}_t^* \left[\frac{Y_t}{\phi \xi} + \frac{\ln F(T-t, Y_t)}{\phi} \right] . \end{aligned} $$$$$

In particular, at $(t=0)$ we obtain $(\psi_0^* = w \left[\frac{y}{\phi \xi} + \frac{\ln F(T,y)}{\phi} \right])$ as in the statement of the proposition. (\square)

Rights and permissions

[Reprints and permissions](#)

About this article

Cite this article

Battauz, A., De Donno, M. & Sbuelz, A. Reaching nirvana with a defaultable asset?. *Decisions Econ Finan* **40**, 31–52 (2017). <https://doi.org/10.1007/s10203-017-0192-x>

Received

07 October 2016

Issue date

November 2017

DOI

<https://doi.org/10.1007/s10203-017-0192-x>

Accepted

18 May 2017

Published

01 June 2017

Keywords

[Dynamic asset allocation](#)

[Duality-based optimal portfolio solutions](#)

[Convex duality](#)

[Non-myopic speculation](#)

[Leverage](#)

[Investment horizon](#)

[Sharpe ratio risk](#)

[Reaching for yield](#)

[Predictable default risk](#)

JEL Classification

[G01](#)

[G10](#)

[G11](#)

[G12](#)

[D84](#)

[D90](#)

[C61](#)

Profiles

1. Anna Battauz



[View author profile](#)

2. Marzia De Donno



[View author profile](#)

Search

Search by keyword or author



Navigation

[Find a journal](#)

[Publish with us](#)

[Track your research](#)

