# Reaching nirvana with a defaultable asset?

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## Notes

1. Cox and Huang (1989) require the global Lipschitz continuity of the diffusive coefficient for the risky asset-value process [see Conditions A and B at p. 46 in Cox and Huang (1989)]. By contrast, the diffusive coefficient in our setting is

6. There are parameter values that make the investor with $\phi >1$ take a net short position in the risky defaultable asset.

7. Similar investment-horizon effects have been found in the literature on dynamic portfolio choice with a risky non-defaultable asset characterized by a mean-reverting drift and a constant volatility [see, for example, Koijen et al. (2009)].

8. The functions $U$ and $V$ are conjugate if and only if $U(w)=\inf _{y>0}(V(y)+wy)$ and $V(y)=\sup _{w>0}(U(w)-wy)$.

9. The superscript $\left( \cdot \right) ^{DS}$ refers to the notations of Delbaen

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## 1.1 A.1 Proof of proposition 3.1

The problem of utility maximization can be written as $J(w)=\left( e^{rT}\right) ^{1-\phi }u(w)$ where *u* is defined as

$$\begin{aligned} u(w)=\sup _{{\tilde{W}}\in {\tilde{\mathcal {W}}}(w)}E[U({\tilde{W}}_{T})]. \end{aligned}$$

(A.1)

We apply to problem (A.1) the duality approach developed by Kramkov and Schachermayer (1999, 2003). To this aim, we observe that the utility function *U* satisfies Inada conditions [equation (2.4) in Kramkov and Schachermayer (1999)]. Let *V* denote the conjugate function[8] of *U*, that is $V(y)=\frac{\phi }{1-\phi }y^{-\frac{1-\phi }{\phi }},$ and define

\frac{1}{\phi }} =w\frac{\eta ^{-\frac{1}{\phi }}}{E\left[ \eta ^{-\frac{1-\phi }{\phi } }\right] }. \end{aligned}$$

The value function is then

$$\begin{aligned} J(w)&=\left( e^{rT}\right) ^{1-\phi }E\left[ U({\tilde{W}}^{*})\right] =U(we^{rT})E\left[ \left( \frac{\eta ^{-\frac{1}{\phi }}}{E\left[ \eta ^{-\frac{1-\phi }{\phi }}\right] }\right) ^{1-\phi }\right] \\&=U(we^{rT})\, \left( {E\left[ \eta ^{-\frac{1-\phi }{\phi }}\right] }\right) ^{\phi }\\&=U(we^{rT})\left( E\left[ \exp \left( \frac{1-\phi }{\phi }\int _{0}^{T}\sqrt{Y_{t}}dZ_{t}+\frac{1-\phi }{2\phi }\int _{0}^{T}Y_{t}dt\right) \right] \right) ^{\phi }\\&=U(we^{rT})F(T,y). \end{aligned}$$

\(\square \)

The random variable $L_{T}$ in (A.2) is the Radon–Nikodym density of a probability measure equivalent to ${\mathbb {P}}$. In fact, Theorem[9] 2.3 in Delbaen and Shirakawa (2002) applied to $S^{DS}=Y,$ $\rho ^{DS}=0.5,$ $\,r^{DS} =b>0,\ \sigma ^{DS}=2\sqrt{c\phi },$ $\mu ^{DS}=b+2a\sqrt{c\phi }$, and $\theta ^{DS}=a$ implies that $\eta _{T}^{DS}=L_{T}$ is the Radon–Nikodym density of an equivalent probability measure ${\hat{\mathbb {Q}}}$ equivalent to ${\mathbb {P}}.$ Girsanov's theorem implies then that

$$\begin{aligned} \hat{Z}_{t}=Z_{t}-\int _{0}^{t}a\sqrt{Y_{s}}ds \end{aligned}$$

(A.3)

is a ${\hat{\mathbb {Q}}}$-Brownian motion. Thus,

(A.6)

where $G(t, y)$ is a $\mathcal {C}^{1,2}$ function to be determined in such a way that $G(0,y)=1$ and $M$ is a ${\hat{\mathbb {Q}}}$-martingale. In particular, Eq. (A.6) implies for $t=0$ that $M_{0}=G(T,y)$ and for $t=T$

$$\begin{aligned} M_{T}=\exp \left( \frac{a^{2}+a}{2}\int _{0}^{T}Y_{t}dt\right) G(0,Y_{T} )=\exp \left( \frac{a^{2}+a}{2}\int _{0}^{T}Y_{t}dt\right) , \end{aligned}$$

since $G(0,\cdot )=1.$ The martingality condition $M_{0}=E^{{\hat{\mathbb {Q}}} }[M_{T}]$ yields then

$$G(T,y)=E^{{\hat{\mathbb {Q}}}}\left[ \exp \left( \dots \right. \right.$$

$$\begin{aligned} g\left( t\right) =\frac{(a^{2}+a)(e^{\sqrt{q}t}-1)}{\sqrt{q}+b+e^{\sqrt{q} t}\left( \sqrt{q}-b\right) }. \end{aligned}$$

Since $M_{t}$ has 0 drift in the Ito decomposition under ${\hat{\mathbb {Q}}}$, $M_{t}$ is a ${\hat{\mathbb {Q}}}$ local martingale. To conclude that $M_{t}$ is a martingale we define

$$\begin{aligned} \mathfrak {z}_{t}=\frac{M_{t}}{M_{0}}, \end{aligned}$$

which is a ${\hat{\mathbb {Q}}}$ *local* martingale as well, and show that $\mathfrak {z}_{t}$ is a ${\hat{\mathbb {Q}}}$ martingale. To this aim, we first observe that process $\mathfrak {z}_{t}$ is a stochastic exponential. In fact, Ito formula implies that

$$\begin{aligned} dM_{t}=e^{\frac{a^{2}+a}{2}\int

with

$$\begin{aligned} m_{t}=\int _{0}^{t}2\sqrt{c\phi }g(T-s)\sqrt{Y_{s}}d\hat{Z}_{s}. \end{aligned}$$

We apply Theorem 4.1 in Klebaner and Lipster (2014) to conclude that $\mathfrak {z}_{t}$ is a true martingale. In particular, with Klebaner and Lipster notations (4.2) at page 44

$$\begin{aligned} a_{s}(y)&=by\quad b_{s}(y)=2\sqrt{c\phi }\sqrt{y}\qquad \text {from }\left( A.5\right) \qquad \text {and}\\ \sigma _{s}(y)&=2\sqrt{c\phi }g(T-s)\sqrt{y}\qquad \qquad \text {from our def. of }m_{t}, \end{aligned}$$

we get

## 1.3 A.3 Proof of proposition 3.3

In what follows, we will mainly work under the martingale measure $\mathbb {Q},$ whose density with respect to ${\mathbb {P}}$ is $\eta$ in Eq. (2.3). We denote with $Z_{t}^{\mathbb {Q}}$ the $\mathbb {Q}$-Brownian motion

$$\begin{aligned} Z_{t}^{\mathbb {Q}}=Z_{t}+\int _{0}^{t}\sqrt{Y_{s}}ds. \end{aligned}$$

(A.10)

Before proving the result, we first list some technical lemmas.

### Lemma A.1

Let $L^{*}=\frac{L_{T}}{\eta }$ where $L_{T}$ is given by (A.2). Then \

{Q}}}}{d{\mathbb {P}}}}{\frac{d\mathbb {Q}}{d{\mathbb {P}}}}=\frac{d{\hat{\mathbb {Q}}}}{d\mathbb {Q}}, \end{aligned}$$

and from the definitions of $\eta$ in (2.3) and of $L$ in (A.2) that

$$\begin{aligned} L_{T}^{*}&=\frac{L_{T}}{\eta }=\exp \left( a\int _{0}^{T}\sqrt{Y_{s} }dZ_{s}-\frac{a^{2}}{2}\int _{0}^{T}Y_{s}ds+\int _{0}^{T}\sqrt{Y_{s}} dZ_{s}+\frac{1}{2}\int _{0}^{T}Y_{s}ds\right) \\&=\exp \left( \left( a+1\right) \int _{0}^{T}\sqrt{Y_{s}}dZ_{s} +\frac{1-a^{2}}{2}\int _{0}^{T}Y_{s}ds\right) . \end{aligned}$$

From the definition of $Z_{s}^{\mathbb {Q}}$ in Eq. (A.10) we get

$$\begin{aligned} L_{T}^{*}&=\exp \left( \left( a+1\right) \int _{0}^{T}\sqrt{Y_{s} }\left( dZ_{s}^{\mathbb {Q}}-\sqrt{Y_{s}}ds\right)

Let $M_{t}$ be the ${\hat{\mathbb {Q}}}$-martingale defined in (A.6), namely

$$\begin{aligned} M_{t}=e^{\frac{a^{2}+a}{2}\int _{0}^{t}Y_{s}ds}e^{Y_{t}g(T-t)}. \end{aligned}$$

Then we have

$$\begin{aligned} dM_{t}&=2\sqrt{c\phi Y_{t}}M_{t}g(T-t)d\hat{Z}_{t}\\&=2\sqrt{c\phi Y_{t}}M_{t}g(T-t)\left( dZ_{t}^{\mathbb {Q}}-\frac{1}{\phi }\sqrt{Y_{t}}dt\right) \nonumber \end{aligned}$$

(A.12)

where $\hat{Z}_{t}$ is defined in (A.3) and $Z_{t}^{\mathbb {Q}}$ in Eq. (A.10).

_{t}^{*}}{\sqrt{Y_{t}}}dZ_{t}^{\mathbb {Q}} \end{aligned}$$

(A.13)

since $d{\tilde{P}}_{t}={\tilde{P}}_{t}\sqrt{Y_{t}}\sigma _{t}dt+{\tilde{P}}_{t}\sigma _{t}dZ_{t}={\tilde{P}}_{t}\frac{\xi }{\sqrt{Y_{t}}}dZ_{t}^{\mathbb {Q}}$ from (2.1) and (A.10).

Therefore, we look for the Ito representation of ${\tilde{W}}^{*}(t)=E^{\mathbb {Q}}[{\tilde{W}}^{*}|\mathcal {F}_{t}]$ to derive $\psi ^{*}$. Denoting with $L_{t}={E\left[ L_{T}\Big \vert \mathcal {F}_{t}\right] },$ and $\eta _{t}={E\left[ \eta \Big \vert \mathcal {F}_{t}\right] }$, we have

$$\begin{aligned} E^{\mathbb {Q}}\left[ \eta ^{-\frac{1}{\phi }}\Big \vert \mathcal {F}_{t}\right]&=\frac{E\left[ \eta ^{1-\frac{1}{\phi }}\Big \vert

{F}_{t}\right] \\&=\frac{w}{E\left[ \eta ^{-\frac{1-\phi }{\phi }}\right] }L_{t}^{*} M_{t}\\&=\frac{w}{G(T,y)}L_{t}^{*}M_{t} \end{aligned}$$

because \(G(T,y)=\left( F(T,y)\right) ^{\frac{1}{\phi }}={E\left[ \eta ^{-\frac{1-\phi }{\phi }}\right] .}\) It follows that the differential of the \(\mathbb {Q}\)-martingale \({\tilde{W}}_{t}^{*}\) is given by

$$\begin{aligned} d{\tilde{W}}_{t}^{*}&=\frac{w}{G(T,y)}d\left( L_{t}^{*}M_{t}\right) \\&=\frac{w}{G(T,y)}L_{t}^{*}M_{t}\left[ \frac{1}{\phi }+2\sqrt{c\phi }g(T-t)\right] \sqrt{Y_{t}}dZ_{t}^{\mathbb {Q}}\quad \hbox {by}\left( { A}.11\right) \text { and }\left( { A}.12\right) \\&= {\tilde{W}}_{t}^{*}\left[ \frac{1}{\phi }+2\sqrt{c\phi }g(T-t)\right] \sqrt{Y_{t}}dZ_{t}^{\mathbb {Q}} \end{aligned}$$

Comparing this equation with Eq. (A.13), we obtain

## About this article

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