Reaching nirvana with a defaultable asset?

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Abstract

We study the optimal dynamic portfolio exposure to predictable default risk, taking inspiration from the search for yield by means of defaultable assets observed before the 2007–2008 crisis and in its aftermath. Under no arbitrage, default risk is compensated by an 'yield pickup' that can strongly attract aggressive investors via an investment-horizon effect in their optimal non-myopic portfolios. We show it by stating the optimal dynamic portfolio problem of Kim and Omberg (Rev Financ Stud 9:141–161, 1996) for a defaultable risky asset and by rigorously proving the existence of nirvana-type solutions. We achieve such a contribution to the portfolio optimization literature by means of a careful, closed-form-yielding adaptation to our defaultable asset setting of the general convex duality approach of Kramkov and Schachermayer (Ann Appl Probab 9(3):904–950, 1999; Ann Appl Probab 13(4):1504–1516, 2003).

Notes

1. Cox and Huang (1989) require the global Lipschitz continuity of the diffusive coefficient for the risky asset-value process [see Conditions A and B at p. 46 in Cox and Huang (1989)]. By contrast, the diffusive coefficient in our setting is the square root of the risky asset value.

2. See, for instance, Davydov and Linetsky (2001), p. 952, first paragraph, with $S_{t}=P_{t}$ and $p=\frac{1}{2}.$

3. The objective probability of the asset defaulting within the date $T>0$ is ${\mathbb {P}}\left[P_{h}=0,\,0\le h\le T\quad P_{0}=p\,\right] =\Gamma \left(\frac{2\left(r+\xi \right) p}{1-e^{-\left(r+\xi \right) T}},1\right) ,$ where $\Gamma \left(k,l\right) =\int _{k}^{+\infty }u^{l-1}e^{-u}du,$ $k\ge 0$ is the incomplete gamma function [see, e.g., the Proposition 1 in Campi and Sbuelz (2005)].

4. The nonnegativity requirement is innocuous in our setting as the utility function *U* is $-\infty$ for negative wealth levels.

5. Battauz et al. (2015) apply the Kramkov and Schachermayer (1999, 2003) approach to the standard Kim and Omberg (1996) portfolio problem with a non-defaultable risky asset.

6. There are parameter values that make the investor with $\phi >1$ take a net short position in the risky defaultable asset.

7. Similar investment-horizon effects have been found in the literature on dynamic portfolio choice with a risky non-defaultable asset characterized by a mean-reverting drift and a constant volatility [see, for example, Koijen et al. (2009)].

8. The functions *U* and *V* are conjugate if and only if $U(w)=\inf _{y>0}(V(y)+wy)$ and $V(y)=\sup _{w>0}(U(w)-wy)$.

9. The superscript $\left(\cdot \right) ^{DS}$ refers to the notations of Delbaen and Shirakawa (2002), whereas $a,\ b,\ c$ are defined in the statement of Proposition 3.2.

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A Appendix

1.1 A.1 Proof of proposition 3.1

The problem of utility maximization can be written as $J(w)=\left(e^{rT}\right)^{1-\phi }u(w)$ where *u* is defined as

$$\begin{aligned} u(w)=\sup _{{\tilde{W}}\in {\tilde{\mathcal {W}}}(w)}E[U({\tilde{W}}_{T})]. \end{aligned}$$

(A.1)

We apply to problem (A.1) the duality approach developed by Kramkov and Schachermayer (1999, 2003). To this aim, we observe that the utility function *U* satisfies Inada conditions [equation (2.4) in Kramkov and Schachermayer (1999)]. Let *V* denote the conjugate function[8] of *U*, that is $V(y)=\frac{\phi }{1-\phi }y^{-\frac{1-\phi }{\phi }},$ and define

$$\begin{aligned} v(y)=E\left[V\left(y\eta \right) \right] \end{aligned}$$

where η is given by (2.3). Kramkov and Schachermayer (2003) show that if $v(y)<\infty$ for all $y>0$, then $u(w)<\infty$ for all $w>0$ and *u* and *v* are conjugate. They also prove that the optimal solution ${\tilde{W}}^{*}\in {\tilde{W}}(w)$ to (A.1) exists and is unique. Moreover, taking $y=u^{\prime }(w)$ (or equivalently $w=-v^{\prime }(y)$), they provide the dual relation for the optimizer $\tilde{W^{*}}=-V^{\prime }(y\eta)$ (see Theorems 1,2 and Note 3).

Assuming that $E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] <+\infty$, from the condition $w=-v^{\prime }(y)$, we get

$$\begin{aligned} y=\left(\frac{w}{E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] }\right) ^{-\phi } \end{aligned}$$

and

$$\begin{aligned} {\tilde{W}}^{*} =-V^{\prime }(y\eta)=\left(\left(\frac{w}{E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] }\right) ^{-\phi }\eta \right) ^{-\frac{1}{\phi }} =w\frac{\eta ^{-\frac{1}{\phi }}}{E\left[\eta ^{-\frac{1-\phi }{\phi } }\right] }. \end{aligned}$$

The value function is then

$$\begin{aligned} J(w)&=\left(e^{rT}\right) ^{1-\phi }E\left[U({\tilde{W}}^{*})\right] =U(we^{rT})E\left[\left(\frac{\eta ^{-\frac{1}{\phi }}}{E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] }\right) ^{1-\phi }\right] \\&=U(we^{rT})\, \left({E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] }\right) ^{\phi }\\&=U(we^{rT})\left(E\left[\exp \left(\frac{1-\phi }{\phi }\int _{0}^{T}\sqrt{Y_{t}}dZ_{t}+\frac{1-\phi }{2\phi }\int _{0}^{T}Y_{t}dt\right) \right] \right) ^{\phi }\\&=U(we^{rT})F(T,y). \end{aligned}$$

$\square$

1.2 A.2 Proof of proposition 3.2

Since $a=\frac{1-\phi }{\phi },$ then

$$\begin{aligned}&E\left[\exp \left(\frac{1-\phi }{\phi }\int _{0}^{T}\sqrt{Y_{t}}dZ_{t} +\frac{1-\phi }{2\phi }\int _{0}^{T}Y_{t}dt\right) \right] \\&\quad =E\left[\exp \left(a\int _{0}^{T}\sqrt{Y_{t}}dZ_{t}+\frac{a}{2}\int _{0}^{T} Y_{t}dt\right) \right] \end{aligned}$$

We can write:

$$\begin{aligned} E\left[\exp \left(a\int _{0}^{T}\sqrt{Y_{t}}dZ_{t}+\frac{a}{2}\int _{0} ^{T}Y_{t}dt\right) \right] =E\left[L_{T}\exp \left(\frac{a^{2}+a}{2} \int _{0}^{T}Y_{t}dt\right) \right] \end{aligned}$$

where

$$\begin{aligned} L_{T}=\exp \left(a\int _{0}^{T}\sqrt{Y_{t}}dZ_{t} -\frac{a^{2}}{2}\int _{0} ^{T}Y_{t}dt\right) . \end{aligned}$$

(A.2)

The random variable L_{T} in (A.2) is the Radon–Nikodym density of a probability measure equivalent to ${\mathbb {P}}$. In fact, Theorem[9] 2.3 in Delbaen and Shirakawa (2002) applied to $S^{DS}=Y,$ $\rho ^{DS}=0.5,$ $\,r^{DS} =b>0,\ \sigma ^{DS}=2\sqrt{c\phi },$ $\mu ^{DS}=b+2a\sqrt{c\phi }$, and $\theta ^{DS}=a$ implies that $\eta _{T}^{DS}=L_{T}$ is the Radon–Nikodym density of an equivalent probability measure ${\hat{\mathbb {Q}}}$ equivalent to ${\mathbb {P}}.$ Girsanov's theorem implies then that

$$\begin{aligned} \hat{Z}_{t}=Z_{t}-\int _{0}^{t}a\sqrt{Y_{s}}ds \end{aligned}$$

(A.3)

is a ${\hat{\mathbb {Q}}}$-Brownian motion. Thus,

$$\begin{aligned} F(T,y)=\left(E^{{\hat{\mathbb {Q}}}}\left[\exp \left(

$$\frac{a^{2}+a}{2} \int _{0}^{T}Y_{t}dt\right) \right] \right) ^{\phi },
\end{aligned}$$

(A.4)

where Y_{t} has the following dynamics under ${\hat{\mathbb {Q}}}$

$$\begin{aligned} dY_{t}=bY_{t}dt+2\sqrt{c\phi Y_{t}}d\hat{Z}_{t},
\end{aligned}$$

(A.5)

with $Y_{0}=y$. We specify that, if we define the default time $\tau _{y} =\inf \{t\ge 0:Y_{t}=0\}$, we have $Y_{t}\equiv 0$ on $\{\tau _{y}\le t\}$ and *Y* satisfies the stochastic differential Eq. (A.5) on the whole time interval $\left[0,T\right]$. To compute the expectation in (A.4), we define the process

$$\begin{aligned} M_{t}=\exp \left(\frac{a^{2}+a}{2}\int _{0}^{t}Y_{s}ds\right) G(T-t,Y_{t}) \end{aligned}$$

(A.6)

where *G*(*t*, *y*) is a $\mathcal {C}^{1,2}$ function to be determined in such a way that $G(0,y)=1$ and *M* is a ${\hat{\mathbb {Q}}}$-martingale. In particular, Eq. (A.6) implies for $t=0$ that $M_{0}=G(T,y)$ and for $t=T$

$$\begin{aligned} M_{T}=\exp \left(\frac{a^{2}+a}{2}\int _{0}^{T}Y_{t}dt\right) G(0,Y_{T})=\exp \left(\frac{a^{2}+a}{2}\int _{0}^{T}Y_{t}dt\right) , \end{aligned}$$

since $G(0,\cdot)=1.$ The martingality condition $M_{0}=E^{{\hat{\mathbb {Q}}} }[M_{T}]$ yields then

$$\begin{aligned} G(T,y)=E^{{\hat{\mathbb {Q}}}}\left[\exp \left(\frac{a^{2}+a}{2}\int _{0} ^{T}Y_{s}dt\right) \right] , \end{aligned}$$

that allows us to find (A.4). By imposing 0 drift on the Ito decomposition under ${\hat{\mathbb {Q}}}$ of the process M_{t} we get the partial differential equation for *G*

$$\begin{aligned} \left\{ \begin{array} [c]{l} G_{t}=\frac{\xi ^{2}}{2}yG_{yy}+byG_{y}+\frac{a^{2}+a}{2}yG\\ G(0,y)=1. \end{array} \right. \end{aligned}$$

(A.7)

We guess a solution of the form $G(t,y)=e^{yg(t)}$ and we obtain the following differential equation for *g*:

$$\begin{aligned} \left\{ \begin{array} [c]{l} g^{\prime }(t)=\frac{\xi ^{2}}{2}g^{2}(t)+bg(t)+\frac{a^{2}+a}{2}\\ g(0)=0. \end{array} \right. \end{aligned}$$

(A.8)

Equation (A.8) is a Riccati equation whose solution is

$$\begin{aligned} g\left(t\right) =\frac{(a^{2}+a)(e^{\sqrt{q}t}-1)}{\sqrt{q}+b+e^{\sqrt{q} t}\left(\sqrt{q}-b\right) }. \end{aligned}$$

Since M_{t} has 0 drift in the Ito decomposition under ${\hat{\mathbb {Q}}}$, M_{t} is a ${\hat{\mathbb {Q}}}$ local martingale. To conclude that M_{t} is a martingale we define

$$\begin{aligned} \mathfrak {z}_{t}=\frac{M_{t}}{M_{0}}, \end{aligned}$$

which is a ${\hat{\mathbb {Q}}}$ *local* martingale as well, and show that $\mathfrak {z}_{t}$ is a ${\hat{\mathbb {Q}}}$ martingale. To this aim, we first observe that process $\mathfrak {z}_{t}$ is a stochastic exponential. In fact, Ito formula implies that

$$\begin{aligned} dM_{t}=e^{\frac{a^{2}+a}{2}\int

_{0}^{t}Y_{s}ds}\frac{\partial }{\partial y}G(T-t,Y_{t})2\sqrt{c\phi Y_{t}}d\hat{Z}_{t} \end{aligned}$$

from the dynamics of Y with respect to $\hat{\mathbb {Q}}$ in Eq. (A.5). Since $\frac{\partial }{\partial y} G(T-t,Y_{t})=e^{Y_{t}g(T-t)}\cdot g(T-t)=G(T-t,Y_{t})\cdot g(T-t),$ we obtain

$$\begin{aligned} dM_{t}=e^{\frac{a^{2}+a}{2}\int _{0}^{t}Y_{s}ds}G(T-t,Y_{t})\cdot g(T-t)2\sqrt{c\phi Y_{t}}d\hat{Z}_{t} \end{aligned}$$

leading to

$$\begin{aligned} dM_{t}=2\sqrt{c\phi Y_{t}}M_{t}g(T-t)d\hat{Z}_{t}. \end{aligned}$$

(A.9)

Therefore,

$$\begin{aligned} \mathfrak {z}_{t}=1+\int _{0}^{t}\mathfrak {z}_{s}dm_{s} \end{aligned}$$

with

$$\begin{aligned} m_{t}=\int _{0}^{t}2\sqrt{c\phi }g(T-s)\sqrt{Y_{s}}d\hat{Z}_{s}. \end{aligned}$$

We apply Theorem 4.1 in Klebaner and Lipster (2014) to conclude that $\mathfrak {z}_{t}$ is a true martingale. In particular, with Klebaner and Lipster notations (4.2) at page 44

$$\begin{aligned} a_{s}(y)&=by\quad b_{s}(y)=2\sqrt{c\phi }\sqrt{y}\qquad \text {from }\left(A.5\right) \qquad \text {and}\\ \sigma _{s}(y)&=2\sqrt{c\phi }g(T-s)\sqrt{y}\qquad \qquad \text {from our def. of }m_{t}, \end{aligned}$$

we get

$$\begin{aligned} L_{s}(y)&=2ya_{s}(y)+\left(b_{s}(y)\right) ^{2}\\&=2by^{2}+4c\phi y\ \\ \mathcal {L}_{s}(y)&=2y\left[a_{s}(y)+b_{s}(y)\sigma _{s}(y)+\left(b_{s}(y)\right) ^{2}\right] \\&=2by^{2}+8c\phi y^{2}g(T-s)+8c\phi y^{2} \end{aligned}$$

Since g is bounded, it follows that $\left(\sigma _{s}(y)\right) ^{2},$ $L_{s}(y),$ and $\mathcal {L}_{s}(y)$ are all dominated by a quadratic polynomial in y, and therefore, assumptions (1)-(2)-(3) of Theorem 4.1 are satisfied. This allows us to conclude that $\mathfrak {z}_{t}=\frac{M_{t} }{M_{0}}$ is a martingale and therefore M_{t} is a martingale as well. Hence we can write (A.4) as

$$\begin{aligned} F(T,y)&=\left(\exp \left(y\frac{(a^{2}+a)(e^{\sqrt{q}T}-1)}{\sqrt{q}+b+e^{\sqrt{q}T}\left(\sqrt{q}-b\right) }\right) \right) ^{\phi }\\&=\exp \left(y\frac{a(e^{\sqrt{q}T}-1)}{\sqrt{q}+b+e^{\sqrt{q}T}\left(\sqrt{q}-b\right) }\right) , \end{aligned}$$

since

$$\begin{aligned} \phi (a^{2}+a)=a. \end{aligned}$$

$\square$

1.3 A.3 Proof of proposition 3.3

In what follows, we will mainly work under the martingale measure $\mathbb {Q},$ whose density with respect to ${\mathbb {P}}$ is η in Eq. (2.3). We denote with $Z_{t}^{\mathbb {Q}}$ the $\mathbb {Q}$-Brownian motion

$$\begin{aligned} Z_{t}^{\mathbb {Q}}=Z_{t}+\int _{0}^{t}\sqrt{Y_{s}}ds. \end{aligned} \tag{A.10}$$

Before proving the result, we first list some technical lemmas.

Lemma A.1

Let $L^{*}=\frac{L_{T}}{\eta }$ where L_{T} is given by (A.2). Then $L_{t}^{*}=E^{\mathbb {Q}}\left[L^{*}|\mathcal {F}_{t}\right]$ satisfies the stochastic differential equation

$$\begin{aligned} dL_{t}^{*}=(a+1)L_{t}^{*}\sqrt{Y_{t}}dZ_{t}^{\mathbb {Q}}=\frac{1}{\phi }L_{t}^{*}\sqrt{Y_{t}}dZ_{t}^{\mathbb {Q}} \end{aligned}$$

(A.11)

with the initial condition $L_{0}^{*}=1$. In particular, L^{*} is the Radon–Nikodym density of the probability measure ${\hat{\mathbb {Q}}}$ (whose density with respect to ${\mathbb {P}}$ is L in (A.2) with respect to $\mathbb {Q}$ and $\hat{Z}_{t}$ defined in (A.3) can be written as

$$\begin{aligned} \hat{Z}_{t}=Z_{t}^{\mathbb {Q}}-\frac{1}{\phi }\int _{0}^{t}\sqrt{Y_{s}}ds. \end{aligned}$$

Proof

It is easy to observe that

$$\begin{aligned} L^{*}=\frac{L_{T}}{\eta }=\frac{\frac{d{\hat{\mathbb {Q}}}}{d{\mathbb {P}}}}{\frac{d\mathbb {Q}}{d{\mathbb {P}}}}=\frac{d{\hat{\mathbb {Q}}}}{d\mathbb {Q}}, \end{aligned}$$

and from the definitions of η in (2.3) and of L in (A.2) that

$$\begin{aligned} L_{T}^{*}&=\frac{L_{T}}{\eta }=\exp \left(a\int _{0}^{T}\sqrt{Y_{s} }dZ_{s}-\frac{a^{2}}{2}\int _{0}^{T}Y_{s}ds+\int _{0}^{T}\sqrt{Y_{s}} dZ_{s}+\frac{1}{2}\int _{0}^{T}Y_{s}ds\right) \\&=\exp \left(\left(a+1\right) \int _{0}^{T}\sqrt{Y_{s}}dZ_{s} +\frac{1-a^{2}}{2}\int _{0}^{T}Y_{s}ds\right) . \end{aligned}$$

From the definition of $Z_{s}^{\mathbb {Q}}$ in Eq. (A.10) we get

$$\begin{aligned} L_{T}^{*}&=\exp \left(\left(a+1\right) \int _{0}^{T}\sqrt{Y_{s} }\left(dZ_{s}^{\mathbb {Q}}-\sqrt{Y_{s}}ds\right) +\frac{1-a^{2}}{2}\int _{0}^{T}Y_{s}ds\right) \\&=\exp \left(\left(a+1\right) \int _{0}^{T}\sqrt{Y_{s}}dZ_{s} ^{\mathbb {Q}}-\frac{\left(a+1\right) ^{2}}{2}\int _{0}^{T}Y_{s}ds\right) . \end{aligned}$$

This is equivalent to

$$\begin{aligned} dL_{t}^{*}=(a+1)L_{t}^{*}\sqrt{Y_{t}}dZ_{t}^{\mathbb {Q}}. \end{aligned}$$

Moreover, from the definition of $\hat{Z}_{t}$ in (A.3) we get

$$\begin{aligned} \hat{Z}_{t}&=Z_{t}-\int _{0}^{t}a\sqrt{Y_{s}}ds\\&=Z_{t}^{\mathbb {Q}}-\int _{0}^{t}\sqrt{Y_{s}}ds-\int _{0}^{t}a\sqrt{Y_{s} }ds \quad (\hbox {from }\left({ A}.10\right))\\&=Z_{t}^{\mathbb {Q}}-\left(a+1\right) \int _{0}^{t}\sqrt{Y_{s}}ds \quad \left(\hbox {with } a+1=\frac{1}{\phi }\right) , \end{aligned}$$

that proves the lemma. $\square$

Lemma A.2

Let M_{t} be the ${\hat{\mathbb {Q}}}$-martingale defined in (A.6), namely

$$\begin{aligned} M_{t}=e^{\frac{a^{2}+a}{2}\int _{0}^{t}Y_{s}ds}e^{Y_{t}g(T-t)}. \end{aligned}$$

Then we have

$$\begin{aligned} dM_{t}&=2\sqrt{c\phi Y_{t}}M_{t}g(T-t)d\hat{Z}_{t}\\&=2\sqrt{c\phi Y_{t}}M_{t}g(T-t)\left(dZ_{t}^{\mathbb {Q}}-\frac{1}{\phi }\sqrt{Y_{t}}dt\right) \nonumber \end{aligned}$$

(A.12)

where $\hat{Z}_{t}$ is defined in (A.3) and $Z_{t}^{\mathbb {Q}}$ in Eq. (A.10).

Proof of the Lemma

The first line in Eq. (A.12) is Eq. (A.9), that leads to (A.12) by recalling that

$$\begin{aligned} \hat{Z}_{t}=Z_{t}^{\mathbb {Q}}-\frac{1}{\phi }\int _{0}^{t}\sqrt{Y_{s}}ds. \end{aligned}$$

$\square$

Proof of proposition 3.3

The discounted optimizer

$$\begin{aligned} {\tilde{W}}^{*}=w\frac{\eta ^{-\frac{1}{\phi }}}{E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] } \end{aligned}$$

is the value at time T of a self-financing discounted portfolio, which admits the following representation under $\mathbb {Q}$

$$\begin{aligned} {\tilde{W}}^{*}=w+\int _{0}^{T}\psi _{t}^{*}\frac{d{\tilde{P}}_{t}}{\tilde{P}_{t}}=w+\int _{0}^{T}\frac{\xi \psi _{t}^{*}}{\sqrt{Y_{t}}}dZ_{t}^{\mathbb {Q}} \end{aligned}$$

(A.13)

since $d{\tilde{P}}_{t}={\tilde{P}}_{t}\sqrt{Y_{t}}\sigma _{t}dt+ {\tilde{P}}_{t}\sigma _{t}dZ_{t}={\tilde{P}}_{t}\frac{\xi } {\sqrt{Y_{t}}}dZ_{t}^{\mathbb {Q}}$ from (2.1) and (A.10).

Therefore, we look for the Ito representation of ${\tilde{W}}^{*} (t)=E^{\mathbb {Q}}[{\tilde{W}}^{*}|\mathcal {F}_{t}]$ to derive $\psi ^{*}$. Denoting with $L_{t}={E\left[L_{T}\Big \vert \mathcal {F}_{t}\right] },$ and $\eta _{t}={E\left[\eta \Big \vert \mathcal {F}_{t}\right] }$, we have

$$\begin{aligned} E^{\mathbb {Q}}\left[\eta ^{-\frac{1}{\phi }}\Big \vert \mathcal {F}_{t}\right]&=\frac{E\left[\eta ^{1-\frac{1}{\phi }}\Big \vert \mathcal {F}_{t}\right] }{\eta _{t}}(\hbox {by Bayes' rule})\\&=\frac{E\left[\exp \left(a\int _{0}^{T}\sqrt{Y_{t}}dZ_{t}+\frac{a}{2} \int _{0}^{T}Y_{t}dt\right) \Big \vert \mathcal {F}_{t}\right] }{\eta _{t}}\\&=\frac{E\left[L_{T}\exp \left(\frac{a^{2}+a}{2}\int _{0}^{T} Y_{t}dt\right) \Big \vert \mathcal {F}_{t}\right] }{\eta _{t}} \qquad (\hbox {by}\left({ A}.2\right))\\&=\frac{L_{t}E^{{\hat{\mathbb {Q}}}}\left[\exp \left(\frac{a^{2}+a}{2} \int _{0}^{T}Y_{t}dt\right) \Big \vert \mathcal {F}_{t}\right] }{\eta _{t}} \qquad (\hbox {by Bayes' rule})\\&=L_{t}^{*}E^{{\hat{\mathbb {Q}}}}\left[\exp \left(\frac{a^{2}+a}{2} \int _{0}^{T}Y_{t}dt\right) \Big \vert \mathcal {F}_{t}\right] \\&=L_{t}^{*}E^{{\hat{\mathbb {Q}}}}\left[M_{T}\Big \vert \mathcal {F} _{t}\right] =L_{t}^{*}M_{t}\qquad (\text {by the definition of }M\text { in }({ A}.6)). \end{aligned}$$

Hence,

$$\begin{aligned} {\tilde{W}}_{t}^{*}&=E^{\mathbb {Q}}[{\tilde{W}}^{*}|\mathcal {F}_{t}]\\&=E^{\mathbb {Q}}\left[\left. w\frac{\eta ^{-\frac{1}{\phi }}}{E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] }\right| \mathcal {F}_{t}\right] \\&=\frac{w}{E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] }E^{\mathbb {Q} }\left[\left. \eta ^{-\frac{1}{\phi }}\right| \mathcal {F}_{t}\right] \\&=\frac{w}{E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] }L_{t}^{*} M_{t}\\&=\frac{w}{G(T,y)}L_{t}^{*}M_{t} \end{aligned}$$

because $G(T,y)=\left(F(T,y)\right) ^{\frac{1}{\phi }}={E\left[\eta ^{-\frac{1-\phi }{\phi }}\right] .}$ It follows that the differential of the $\mathbb {Q}$-martingale ${\tilde{W}}_{t}^{*}$ is given by

$$\begin{aligned} d{\tilde{W}}_{t}^{*}&=\frac{w}{G(T,y)}d\left(L_{t}^{*}M_{t}\right) \\&=\frac{w}{G(T,y)}L_{t}^{*}M_{t}\left[\frac{1}{\phi }+2\sqrt{c\phi }g(T-t)\right] \sqrt{Y_{t}}dZ_{t}^{\mathbb {Q}}\quad \hbox {by}\left({ A}.11\right) \text { and }\left({ A}.12\right) \\&= {\tilde{W}}_{t}^{*}\left[\frac{1}{\phi }+2\sqrt{c\phi }g(T-t)\right] \sqrt{Y_{t}}dZ_{t}^{\mathbb {Q}} \end{aligned}$$

Comparing this equation with Eq. (A.13), we obtain

$$\begin{aligned} \frac{\xi \psi _{t}^{*}}{\sqrt{Y_{t}}}=W^{*}(t)\left[\frac{1}{\phi }+2\sqrt{c\phi }g(T-t)\right] \sqrt{Y_{t}} \end{aligned}$$

hence, recalling that $2\sqrt{c\phi }=\xi$ and $g(T-t)=\frac{1}{Y_{t}}\ln G(T-t,Y_{t})=\frac{1}{\phi Y_{t}}\ln F(T-t,Y_{t})$, we have:

$$\begin{aligned} \psi _{t}^{*}&=\frac{{\tilde{W}}_{t}^{*}}{\xi }\left[\frac{1}{\phi }+2\sqrt{c\phi }g(T-t)\right] Y_{t}\\&={\tilde{W}}_{t}^{*}\left[\frac{Y_{t}}{\phi \xi }+\frac{\ln F(T-t,Y_{t})}{\phi }\right] . \end{aligned}$$

In particular, at $t=0$ we obtain $\psi _{0}^{*}=w\left[\frac{y}{\phi \xi }+\frac{\ln F(T,y)}{\phi }\right] ,$ as in the statement of the proposition. $\square$

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