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Option pricing with an illiquid underlying asset market

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Abstract

We examine how price impact in the underlying asset market affects the replication of a European contingent claim. We obtain a generalized Black–Scholes pricing PDE and establish the existence and uniqueness of a classical solution to this PDE. Unlike the case with transaction costs, we prove that replication with price impact is always cheaper than superreplication. Compared to the Black–Scholes case, a trader generally buys more stock and borrows more (shorts and lends more) to replicate a call (put). Furthermore, price impact implies *endogenous* stochastic volatility and an out-of-money option has lower implied volatility than an in-the-money option. This finding has important implications for empirical analysis on *volatility smile*.

Introduction

Most of the option pricing models assume that an option trader cannot affect the underlying asset price in trading the underlying asset to replicate the option payoff, regardless of her trading size. This is reasonable only in a perfectly liquid market. In a market with imperfect liquidity, however, trading does affect the underlying asset price. Indeed, the presence of price impact of investors' trading has been widely documented and extensively analyzed in the literature (see, for example, Chan and Lakonishok, 1995; Keim and Madhavan, 1996, Sharpe et al., 1999, Jorion, 2000). Even for a very liquid market, trading beyond the quoted depth usually results in a worse price for at least part of the trade.

Consistent with the above discussion, in this paper we take this imperfect liquidity as given and examine how it affects the replication of a European option by a typical option trader. In particular, we assume that as the trader buys the underlying asset (stock) to hedge her position in the option, the stock price goes up and as she sells, the price goes down.

Several issues are critical for understanding how this price impact affects the replication of a European option. First, because in the presence of this adverse price impact trading in the stock to replicate the option affects the stock price, it is not clear whether the option is still perfectly replicable or not. Second, the presence of adverse price impact increases the replicating costs. It is well known that in the presence of transaction cost, superreplication of an option (e.g., buying a share to superreplicate a call) costs less than exact replication. Therefore a natural question is whether it is also cheaper to superreplicate than to exactly replicate in the presence of price impact. Third, if it is cheaper to exactly replicate, what is the extra replication cost over the Black–Scholes price that a trader has to incur? In addition, how should the trader trade the underlying asset to replicate? Finally, what are the implications of the price impact on the well-known *volatility smile*?

To answer these questions, we use the idea of a four-step scheme for forward–backward stochastic differential equations (FBSDEs) (see Ma et al., 1994, Yong, 1999; Yong and Zhou, 1999) to derive a generalized, nonlinear Black–Scholes partial differential equation (PDE) for computing the replicating cost of a European option. We provide sufficient conditions under which the option is perfectly replicable. This pricing PDE shows that the effect of the price impact on the replicating cost is only through the impact of the trader's trading on the stock return volatility. We then show that, unlike the transaction cost case, superreplication is more costly than exact replication. Furthermore, like the Black–Scholes case, the replicating strategy involves an initial block trade followed by continuous trading. In addition, we show that the excess replicating cost over the Black–Scholes price is significant, even with a small price impact in the underlying asset market.

We find that a trader generally buys more stock and borrows more (shorts and lends more) to replicate a call (put). In the special case in which contingent claim payoffs are linear in the stock price (e.g., forwards, futures, or shares), the trader adopts the same strategy as in the case without price impact. However, the cost is higher due to the adverse price impact.

The presence of price impact implies that although a special form of put–call parity still holds, the implied volatility for a put is *different* from the implied volatility for the otherwise identical call. We find that out-of-money options have lower implied volatility than in-the-money options. This pattern is consistent with the well-known volatility smile found for calls, but is different from the volatility smile observed for puts (see Dumas et al., 1998, for example). Intuitively, as a trader trades the price moves against her, so she incurs higher replicating costs. When an option is in the money, she needs to trade more in the stock. So the extra replicating cost over the Black–Scholes price is greater. However, option price is not sensitive to volatility in the Black–Scholes world for an in-the-money option. This implies that a large implied volatility is required to generate the higher replicating costs that resulted from the price impact. When an option is out of the money, however, she needs to trade less in the stock to replicate the option. So the extra cost is smaller and thus the implied volatility is also smaller. The implied volatility patterns we found have important empirical implications for explaining the volatility smile. In particular, it suggests that the negative correlation between the stock price and the volatility (as assumed in Heston, 1993 and Bates, 1996, for example) that was required to generate the smile would have to be weaker for calls, but stronger for puts, if the underlying asset market were illiquid.

A number of option valuation models in the literature attempt to explain the volatility smile. The stochastic volatility models of Heston (1993) and Hull and White (1987), for example, can potentially explain the smile to some extent when the asset price and the volatility are negatively correlated. Similarly, the jump model of Bates (1996) is also consistent with the smile when the mean jump is negative. The deterministic volatility model examined by Dumas et al. (1998) can also generate a similar smile pattern. However, all these models assume an exogenous volatility process. In contrast, in our model the volatility process is endogenous and is affected by the trading of the trader.

In the presence of price impact, the no-arbitrage price of an option for a trader is no longer unique. Rather, it consists of a continuum of prices within an interval. We find that this no-arbitrage interval expands as the price impact increases. In addition, the price impact also introduces nonlinearity into the dynamics of the replicating portfolio value. We show that the excess replicating cost is approximately quadratic in the number of units of an option.

There is an extensive literature on the effect of price impact. In the presence of asymmetric information, Kyle (1985) and Back (1993) use an equilibrium approach to investigate how informed traders reveal information and affect the market price through trading. As shown by Kyle (1985) and Back (1993), equilibrium asset prices are directly affected by the informed trader's trades. Vayanos (2001) studies a dynamic model of a financial market with a large trader who does not have any private information on the asset value but trades only to share risk. He shows that the equilibrium stock price is linear in the investor's order size. These models provide theoretic justifications for the existence, the form, and the direction of the price impact a trader can have on stock prices. In particular, the price impact form used in this paper, which is linear in the trading size, is consistent with the equilibrium price impact forms derived in these models.

Cvitanić and Ma (1996) and Ma and Yong (1999) examine the hedging costs of options for a trader in the presence of price impact. Cuoco and Cvitanić (1998) consider the effect of the price impact on the optimal consumption and investment policy. In these papers, it was assumed that price impact depends only on the total wealth and the position of a trader but not on how she trades.

Our model is closely related to Frey (2000), Frey and Patie (2002), and Bank and Baum (2004). All these papers consider the replicating or superreplicating cost of a European contingent claim in the presence of price impact. However, all of them ignore the initial block trade (and thus the initial extra cost) that is necessary for any replication. We show that ignoring the initial block trade would significantly understate the replicating costs and produce qualitatively misleading conclusions. In addition, Frey (2000) and Frey and Patie (2002) do not show the existence and uniqueness of a solution to the pricing PDE they derived and thus do not show the replicability of an option. Bank and Baum (2004) only provide a martingale characterization of the superreplicating cost under the assumption that the price impact disappears instantaneously and economic analysis is limited. Sircar and Papanicolaou (1998) assume an exogenous demand function for the reference traders and derive a different nonlinear pricing PDE that depends on the exogenous income process of the reference traders and the relative size of the program traders.

To focus on our main objective of understanding the replicating strategy and the replicating cost for an option written on an illiquid asset, we use a partial equilibrium approach in this paper. In particular, we take the price impact function as given. As will be shown, this model provides an economically sensible characterization of the replicating strategy and a reasonable estimate of the replicating cost for a typical trader in a European option market. In addition, it can be justified by many equilibrium models such as the following example, which is similar to the model in Back (1993). Consider an economy in which there are risk-neutral and competitive stock market makers, risk-averse and competitive option market makers, a risk-neutral informed trader, and liquidity investors. The stock market makers trade only in the stock whereas the option market makers trade only in the options. The insider and liquidity traders can trade both the stock and the options. At a future time T, there is to be a public information release on the stock value that will fix the stock price at an exogenous level $\widehat{S}(T)$, which is known in advance only by the informed trader. All options expire immediately after the release of the public information at T. Due to competition and risk aversion, the option market makers trade options at the replicating costs and perfectly hedge in the stock market. In equilibrium, the insider will trade in such a way that the stock price at T will be exactly equal to $\widehat{S}(T)$ given the option market makers' hedging trades and liquidity traders' liquidity trades. The price impact function assumed in the model can be interpreted as the equilibrium price response function to the option market maker's hedging trades, given the liquidity traders' and the insider's equilibrium trades. In this framework, of price manipulation in the stock market, such as that suggested by Jarrow (1992), Allen and Gale (1992), Vila (1989), Bagnoli and Lipman (1990), and Schönbucher and Wilmott (2000), to affect the payoff of an option is impossible because the payoff only depends on $\widehat{S}(T)$ and the option's strike price.

The rest of the paper is organized as follows. In Section 2, we introduce our model. In Section 3, we derive the generalized Black–Scholes pricing PDE in the presence of price impact and provide sufficient conditions under which a European option can be replicated. We also show that superreplication is more expensive than exact replication. In Section 4, we provide a numerical analysis of the model for European calls and puts. Section 5 contains the concluding remarks. In the Appendix, we provide proofs of the theorems.

Section snippets

The model

Throughout this paper we fix a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbf{P})$ on which a standard one-dimensional Brownian motion $B(t)$ is defined with $\{\mathscr{F}_t\}_{t\geqslant 0}$ being its natural filtration augmented by all the $\mathbf{P}$-null sets. All the stochastic processes in this paper are assumed to be $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted.

There are two assets being continuously traded in the primary market. The first asset is a money market account. The second is a risky asset, which we will call a stock. Let $S(t)$ be the...

Replication of a European option

Let $h(S(T))$ be the payoff of a European contingent claim maturing at time T, where $h:\mathbb{R}_+ \to \mathbb{R}$ is a piecewise linear function and $S(T)$ is the price of the stock at time T. Hereafter, for convenience, we simply call $h(S(T))$ an option. We assume that the option trader's objective is to replicate the option for a perfect hedge.[2]...

Analysis of the replication of a European option

In this section, we provide a numerical analysis of the effect of the illiquidity in the underlying asset market on the replication of a European option.

In all the subsequent analysis, we focus on the Black–Scholes economy; that is, all the price coefficients are constant. Unless otherwise stated, we assume the default parameter values to be as follows: the current stock price $S = \$50$, the strike price $K = \$50$, the interest rate $r = 6\%$, the dividend yield $\delta = 0$, the volatility $\sigma = 40\%$, and the time to...

Concluding remarks

In this paper, we investigate how the imperfect liquidity in the underlying asset market affects the replication of a European option. In a market with imperfect liquidity, trading affects stock price. We obtain a generalized nonlinear Black–Scholes pricing partial differential equation. We derive sufficient conditions for the existence and uniqueness of a classical solution. These are also sufficient conditions for the replicability of the option. We also show that in contrast to the case with …

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